

# ON STRONGLY SEPARATELY CONTINUOUS FUNCTIONS ON SEQUENCE SPACES

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**ABSTRACT.** We study strongly separately continuous real-valued function defined on the Banach spaces  $\ell_p$ . Determining sets for the class of strongly separately continuous functions on  $\ell_p$  are characterized. We prove that for every  $1 \leq \alpha < \omega_1$  there exists a strongly separately continuous function which belongs the  $(\alpha+1)$ 'th Baire class and does not belong to the  $\alpha$ 'th Baire class on  $\ell_p$ . We show that any open set in  $\ell_p$  is the set of discontinuities of a strongly separately continuous real-valued function.

## 1. INTRODUCTION

Let  $(X_t : t \in T)$  be a family of sets  $X_t$  with  $|X_t| > 1$  for all  $t \in T$ . For  $S \subseteq T$  we put  $X_S = \prod_{t \in S} X_t$ . If  $S \subseteq S_1 \subseteq T$ ,  $x = (x_t)_{t \in T} \in X_T$ ,  $a = (a_t)_{t \in S_1} \in X_{S_1}$ , then we denote by  $x_S^a$  the point  $(y_t)_{t \in T} \in X_T$  defined by

$$y_t = \begin{cases} a_t, & t \in S, \\ x_t, & t \in T \setminus S. \end{cases}$$

In the case of  $S = \{s\}$  we write  $x_s^a$  instead of  $x_{\{s\}}^a$ .

If  $S \subseteq T$ , then we put  $\pi_S(x) = (x_t)_{t \in S}$ .

For  $n \in \mathbb{N}$  let  $\sigma_n(x) = \{y = (y_t)_{t \in T} \in X_T : |\{t \in T : y_t \neq x_t\}| \leq n\}$  and  $\sigma(x) = \bigcup_{n=1}^{\infty} \sigma_n(x)$ . Each of the sets of the form  $\sigma(x)$  we call a  $\sigma$ -product of  $X_T$ .

A set  $X \subseteq X_T$  is called  $\mathcal{S}$ -open [4] if  $\sigma_1(x) \subseteq X$  for all  $x \in X$ . Notice that the definition of an  $\mathcal{S}$ -open set develops the definition of a set of the type  $(P_1)$  introduced in [1]. Observe that  $\sigma$ -products of two distinct points of  $X_T$  either coincide, or do not intersect. Thus, the family of all  $\sigma$ -products of an arbitrary  $\mathcal{S}$ -open set  $X \subseteq X_T$  generates a partition of  $X$  on mutually disjoint  $\mathcal{S}$ -open sets, which we will call  $\mathcal{S}$ -components of  $X$ .

Let  $X \subseteq X_T$  be an  $\mathcal{S}$ -open set,  $\tau$  be a topology on  $X$  and let  $(Y, d)$  be a metric space. A mapping  $f : (X, \tau) \rightarrow Y$  is called *strongly separately continuous at a point  $a \in X$  with respect to the  $t$ -th variable* if

$$\lim_{x \rightarrow a} d(f(x), f(x_t^a)) = 0.$$

A mapping  $f : X \rightarrow Y$  is *strongly separately continuous at a point  $a \in X$*  if  $f$  is strongly separately continuous at  $a$  with respect to every variable  $t \in T$ ; and  $f$  is *strongly separately continuous on the set  $X$*  if  $f$  is strongly separately continuous at every point  $a \in X$  with respect to every variable  $t \in T$ . Strongly separately continuous functions we will also call *ssc-functions* for short.

The notion of real-valued ssc-function defined on  $\mathbb{R}^n$  was introduced by Omar Dzagnidze [2], who proved that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is strongly separately continuous at  $x^0$  if and only if  $f$  is continuous at  $x^0$ .

In [1] the authors extended the notion of the strong separate continuity to functions defined on the Hilbert space  $\ell_2$  equipped with the norm topology. They proved that there exists a real-valued ssc-function on  $\ell_2$  which is everywhere discontinuous. Determining sets (see Definition 2.1) for the class of all ssc-functions were also studied in [1].

The second named author continued to study properties of strongly separately continuous functions on  $\ell_2$  in [8] and constructed a strongly separately continuous function  $f : \ell_2 \rightarrow \mathbb{R}$  which belongs to the third Baire class and is not quasi-continuous at every point. Moreover, he gave a sufficient condition for a strongly separately continuous function to be continuous on  $\ell_2$  and a sufficient condition for a subset of  $\ell_2$  to be determining in the class of real-valued ssc-functions.

The first named author extended the concept of an ssc-function on any  $\mathcal{S}$ -open subset of a product of topological spaces [4]. A characterization of the set of all points of discontinuity of strongly separately

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continuous functions defined on a  $\sigma$ -product of a sequence of finite-dimensional normed spaces was given in [4]. Further, the Baire classification of ssc-functions defined on the space  $\mathbb{R}^\omega$  equipped with the topology of pointwise convergence was investigated in [5]. Moreover, it was shown in [5] that if  $X$  is a product of normed spaces and  $a \in X$  then for any open set  $G \subseteq \sigma(a)$  there is a strongly separately continuous function  $f : \sigma(a) \rightarrow \mathbb{R}$  such that the discontinuity point set of  $f$  is equal to  $G$ . Strongly separately continuous functions defined on a box-product of topological spaces were considered in [6].

Here we study ssc-functions defined on the spaces  $\ell_p$  with  $1 \leq p < +\infty$  of all sequences  $(x_n)_{n=1}^\infty$  of real numbers for which the series  $\sum_{n=1}^\infty |x_n|^p$  is convergent. In Section 2 we find a necessary and sufficient condition on a subset of  $\ell_p$  to be determining in the class of all real-valued ssc-functions on  $\ell_p$ . In the third section we show that for any ordinal  $\alpha \in [1, \omega_1)$  there exists an  $\mathcal{S}$ -open set  $E \subseteq \ell_p$  which is of the  $\alpha$ 'th additive Borel class and does not belong to the  $\alpha$ 'th multiplicative Borel class. Using this fact we construct a real-valued ssc-function on  $\ell_p$  which belongs to the  $(\alpha + 1)$ 'th Baire class and does not belong to the  $\alpha$ 'th Baire class. In Section 4 we prove that for any open nonempty set  $G \subseteq \ell_p$  and  $1 \leq p < \infty$  there exists a strongly separately continuous function  $f : \ell_p \rightarrow \mathbb{R}$  which is discontinuous exactly on  $G$ .

## 2. DETERMINING SETS FOR STRONGLY SEPARATELY CONTINUOUS FUNCTIONS

Let  $(X, Y)$  be a pair of topological spaces and  $\mathcal{F}(X, Y)$  be a class of mappings between  $X$  and  $Y$ .

**Definition 2.1.** A set  $E \subseteq X$  is called *determining for the class  $\mathcal{F}(X, Y)$*  if for any mappings  $f, g \in \mathcal{F}(X, Y)$  the equality  $f|_E = g|_E$  implies that  $f = g$  on  $X$ .

It is well-known that any everywhere dense subset  $E$  of a topological space  $X$  is determining in the class  $C(X, \mathbb{R})$  of all continuous real-valued functions on  $X$ . The theorem of Sierpiński [7] tells us that any everywhere dense subset  $E \subseteq \mathbb{R}^2$  is determining in the class  $CC(\mathbb{R}^2, \mathbb{R})$  of all separately continuous real-valued functions on  $\mathbb{R}^2$ .

In this section we give necessary and sufficient conditions on a subset of an  $\mathcal{S}$ -open set  $X \subseteq \prod_{n=1}^\infty X_n$  to be determining for the class  $\text{SSC}(X)$  of all strongly separately continuous real-valued functions on  $X$ .

The following two notions were introduced in [4].

**Definition 2.2.** A set  $A \subseteq X_T$  is called *projectively symmetric with respect to a point  $a \in A$*  if  $x_t^a \in A$  for all  $t \in T$  and  $x \in A$ .

**Definition 2.3.** Let  $X \subseteq X_T$  and  $\tau$  be a topology on  $X$ . Then  $(X, \tau)$  is said to be *locally projectively symmetric* if every  $x \in X$  has a base of projectively symmetric neighborhoods with respect to  $x$ .

**Definition 2.4.** Let  $(X_t : t \in T)$  be a family of topological spaces and  $X \subseteq X_T$  be an  $\mathcal{S}$ -open set. We say that a topology  $\tau$  on  $X$  is *finitely generated* if for every  $a \in X$  and every finite set  $S \subseteq T$  the space  $(X_S \times \prod_{t \in T \setminus S} \{a_t\}, \tau)$  is homeomorphic to the space  $X_S$  with the topology of pointwise convergence.

Notice that an arbitrary  $\mathcal{S}$ -open subset of a product  $X_T$  of topological spaces  $X_t$  equipped with the topology of pointwise convergence is a locally projectively symmetric space with a finitely generated topology. All classical spaces of sequences as the space  $c$  of all convergence sequences or the spaces  $\ell_p$  with  $0 < p \leq \infty$  are locally projectively symmetric with a finitely generated topology.

Throughout the paper we consider only finitely generated topologies.

We need the following result [4, Theorem 3.4].

**Theorem 2.1.** Let  $X \subseteq X_T$  be an  $\mathcal{S}$ -open set equipped with a locally projectively symmetric topology  $\tau$  and  $x_0 \in X$ .

- (i) If  $f : X \rightarrow \mathbb{R}$  and  $g : X \rightarrow \mathbb{R}$  are ssc-functions at  $x_0$ , then  $f(x) \pm g(x)$ ,  $f(x) \cdot g(x)$ ,  $|f(x)|$ ,  $\min\{f(x), g(x)\}$  and  $\max\{f(x), g(x)\}$  are ssc-functions at  $x_0$ .
- (ii) If  $(f_n)_{n=1}^\infty$  is a sequence of ssc-functions at the point  $x_0$  and the series  $\sum_{n=1}^\infty f_n(x)$  is convergent uniformly on  $X$ , then the sum  $f(x)$  is an ssc-function at  $x_0$ .
- (iii) If  $(f_i)_{i \in I}$  is a locally finite family of ssc-functions  $f_i : X \rightarrow \mathbb{R}$  at  $x_0$ , then  $f(x) = \sum_{i \in I} f_i(x)$  is an ssc-function at  $x_0$ .

It is worth noting that the quotient of two real-valued ssc-functions need not be an ssc-function [3].

**Definition 2.5.** Let  $(X_t : t \in T)$  be a family of topological spaces  $X_t$  and  $a \in X_T$ . A set  $W \subseteq \sigma(a)$  is called *nearly open in  $\sigma(a)$*  if for any finite set  $T_0 \subseteq T$  the set

$$W_{T_0} = \{x \in X_{T_0} : a_{T_0}^x \in W\}$$

is open in the space  $X_{T_0}$  equipped with the topology of pointwise convergence.

**Definition 2.6.** Let  $(X_t : t \in T)$  be a family of topological spaces  $X_t$ ,  $X \subseteq X_T$  be an  $\mathcal{S}$ -open set and  $(\sigma_i : i \in I)$  be a partition of  $X$  on  $\mathcal{S}$ -components. A set  $W \subseteq X$  is called *nearly open in  $X$*  if  $W \cap \sigma_i$  is nearly open in  $\sigma_i$  for every  $i \in I$ .

**Definition 2.7.** Let  $X \subseteq X_T$  be an  $\mathcal{S}$ -open set. A set  $H \subseteq X$  is called *nearly closed in  $X$*  if the complement  $X \setminus H$  is nearly open in  $X$ .

For a set  $A \subseteq \sigma(a)$  we put

$$\overline{A}^\bullet = \{x \in \sigma(a) : W \cap A \neq \emptyset \quad \forall W - \text{ nearly open, } x \in W\}.$$

**Definition 2.8.** A set  $A \subseteq \sigma(a)$  is said to be *super dense (p-dense)* in  $\sigma(a)$  if  $\overline{A}^\bullet = \sigma(a)$  (respectively,  $\overline{A}^p = \sigma(a)$ ), where  $\overline{A}^p$  stands for the closure of  $A$  in the topology of pointwise convergence on  $\sigma(a)$ .

**Definition 2.9.** A subset  $A$  of an  $\mathcal{S}$ -open set  $X = \bigcup_{i \in I} \sigma_i \subseteq X_T$ , where  $(\sigma_i : i \in I)$  is a family of all  $\mathcal{S}$ -components of  $X$ , is called *super dense in  $X$*  if for every  $i \in I$  the set  $A_i = A \cap \sigma_i$  is super dense in  $\sigma_i$  whenever  $A_i \neq \emptyset$ .

Observe that  $A$  is nearly closed if and only if  $\overline{A}^\bullet = A$ . Notice also that every super dense set is p-dense.

**Theorem 2.2.** [6, Theorem 1] *Let  $(X_t)_{t \in T}$  be a family of topological spaces,  $a \in X_T$ ,  $\tau$  be a topology on  $\sigma(a)$  and  $f : \sigma(a) \rightarrow \mathbb{R}$  be an ssc-function. Then  $f^{-1}(V)$  is nearly open in  $\sigma(a)$  for any open set  $V \subseteq \mathbb{R}$ .*

**Proposition 2.3.** *Let  $(X_t : t \in T)$  be a family of topological spaces and  $E$  be a super dense set in an  $\mathcal{S}$ -open space  $X \subseteq X_T$  equipped with a locally projectively symmetric topology  $\tau$ . Then  $E$  is determining set for the class  $\text{SSC}(X)$ .*

*Proof.* Consider ssc-functions  $f, g : X \rightarrow \mathbb{R}$  such that  $f|_E = g|_E$  and let  $(\sigma_i : i \in I)$  be a partition of  $X$  on mutually disjoint  $\mathcal{S}$ -components. Denote

$$H = \{x \in X : f(x) - g(x) = 0\}$$

and for every  $i \in I$  we put  $E_i = E \cap \sigma_i$  and  $H_i = H \cap \sigma_i$ . Since the function  $h(x) = f(x) - g(x)$  is strongly separately continuous on  $X$  by Theorem 2.1, every set  $H_i = (h^{-1}(0)) \cap \sigma_i$  is nearly closed in  $\sigma_i$  according to Theorem 2.2. Then the inclusion  $E_i \subseteq H_i$  implies that  $\sigma_i = \overline{E_i}^\bullet \subseteq \overline{H_i}^\bullet = H_i$ . Consequently,  $H_i = \sigma_i$  for every  $i \in I$ . Therefore,  $H = \bigcup_{i \in I} H_i = \bigcup_{i \in I} \sigma_i = X$ .  $\square$

**Lemma 2.4.** *Let  $(X_n)_{n=1}^\infty$  be a sequence of locally compact Hausdorff spaces,  $a \in \prod_{n=1}^\infty X_n$  and  $W \subseteq \sigma(a)$  be a nearly open set. Then for every  $x \in W$  there exists a sequence  $(U_n)_{n=1}^\infty$  of functionally open sets  $U_n \subseteq X_n$  such that  $x \in (\prod_{n=1}^\infty U_n) \cap \sigma(a) \subseteq W$ .*

*Proof.* For every  $n \in \mathbb{N}$  we denote  $Y_n = \prod_{k=1}^n X_k$  and  $W_n = W_{\{1, \dots, n\}} = \{x \in Y_n : a_{\{1, \dots, n\}}^x \in W\}$ . Take a number  $N$  such that  $x_n = a_n$  for all  $n > N$ . Notice that  $W_n$  is an open subset of the locally compact Hausdorff (and, consequently, completely regular) space  $Y_n$ . Therefore, for every  $n = 1, \dots, N$  there exists a functionally open neighborhood  $U_n$  of  $x_n$  with compact closure such that  $K_1 = \prod_{n=1}^N \overline{U_n} \subseteq W_N$ . For every  $x \in K_1$  we take a functionally open neighborhood  $V_x \times G_x$  of  $(x_1, \dots, x_N, a_{N+1})$  with compact closure in  $Y_N \times X_{N+1}$  such that  $(x_1, \dots, x_N, a_{N+1}) \in V_x \times G_x \subseteq W_{N+1}$ . Since the set  $K_1 \times \{a_{N+1}\}$  is compact, there exists a finite set  $I \subseteq K_1$  such that  $K_1 \times \{a_{N+1}\} \subseteq \bigcup_{x \in I} (V_x \times G_x)$ . Put  $U_{N+1} = \bigcap_{x \in I} G_x$ . Then  $U_{N+1}$  is a functionally open neighborhood of  $a_{N+1}$  in  $X_{N+1}$  and  $K_2 = K_1 \times \overline{U_{N+1}} \subseteq W_{N+1}$ . Proceeding inductively in this way we obtain a sequence  $(U_n)_{n=1}^\infty$  of functionally open sets  $U_n \subseteq X_n$  with  $x \in (\prod_{n=1}^\infty U_n) \cap \sigma(a) \subseteq W$ .  $\square$

The following result follows from [6, Lemma 2].

**Proposition 2.5.** Let  $(X_n)_{n=1}^\infty$  be a sequence of topological spaces,  $a \in \prod_{n=1}^\infty X_n$ ,  $(U_n)_{n=1}^\infty$  be a sequence of functionally open sets  $U_n \subseteq X_n$ ,  $W = (\prod_{n=1}^\infty U_n) \cap \sigma(a)$  and let  $\tau$  be the topology of pointwise convergence on  $\sigma(a)$ . Then there exists an ssc-function  $f : (\sigma(a), \tau) \rightarrow [0, 1]$  such that  $W = f^{-1}((0, 1])$ .

**Proposition 2.6.** Let  $X \subseteq \prod_{n=1}^\infty X_n$  be an  $\mathcal{S}$ -open subset of the product of a sequence of locally compact Hausdorff spaces  $X_n$ ,  $\mathcal{T}$  is a topology on  $X$  which is finer than the topology  $\tau$  of pointwise convergence and  $E \subseteq X$  be a determining set for the class  $\text{SSC}(X)$ . Then  $E$  is super dense in  $X$ .

*Proof.* Consider a partition  $(\sigma_i : i \in I)$  of  $X$  on mutually disjoint  $\mathcal{S}$ -open components. Assume that  $E$  is not super dense in  $X$ . Then there exists  $i \in I$  such that  $\emptyset \neq \overline{E \cap \sigma_i}^\bullet \neq \sigma_i$ . Since  $W = \sigma_i \setminus \overline{E \cap \sigma_i}^\bullet$  is a nonempty nearly open set in  $\sigma_i$ , by Lemma 2.4 there exists a sequence  $(U_n)_{n=1}^\infty$  of functionally open sets  $U_n \subseteq X_n$  such that  $G = (\prod_{n=1}^\infty U_n) \cap \sigma_i \subseteq W$ . According to Proposition 2.5 there exists an ssc-function  $f : (\sigma_i, \tau) \rightarrow [0, 1]$  such that  $G = f^{-1}((0, 1])$ . Since  $\tau \subseteq \mathcal{T}$ ,  $f$  is strongly separately continuous on  $(\sigma_i, \mathcal{T})$ . Notice that  $f|_{E \cap \sigma_i} = 0$  and  $f(x) > 0$  for every  $x \in G$ , which implies a contradiction, since  $E \cap \sigma_i$  is determining in  $\sigma_i$ .  $\square$

Since every space  $\ell_p$  is an  $\mathcal{S}$ -open subset of a countable product  $\mathbb{R}^\omega$  and the standard topology on  $\ell_p$  is finer than the topology of pointwise convergence, Propositions 2.3 and 2.6 immediately imply the following result.

**Theorem 2.7.** For  $p \in [1, +\infty)$  a set  $E \subseteq \ell_p$  is determining for the class  $\text{SSC}(\ell_p)$  if and only if  $E$  is super dense in  $\ell_p$ .

### 3. BAIRE CLASSIFICATION OF SSC-FUNCTIONS ON $\ell_p$

For  $p \in [1, +\infty)$  and  $x = (x_n)_{n=1}^\infty, y \in \ell_p$  we denote

$$\|x\|_p = \left( \sum_{n=1}^\infty |x_n|^p \right)^{\frac{1}{p}} \quad \text{and} \quad d_p(x, y) = \|x - y\|_p.$$

Let  $\pi_n(x) = x_n$  for every  $x = (x_n)_{n=1}^\infty \in \ell_p$  and  $n \in \mathbb{N}$ .

**Lemma 3.1.** For any  $\alpha \in [1, \omega_1)$  and  $p \in [1, +\infty)$  there exists an  $\mathcal{S}$ -open set  $E$  in  $\ell_p$  such that  $E$  belongs to the  $\alpha$ 'th additive class and does not belong to the  $\alpha$ 'th multiplicative class.

*Proof.* Fix  $p \in [1, +\infty)$ .

We define inductively sequences  $(\tilde{A}_\alpha)_{1 \leq \alpha < \omega_1}$  and  $(\tilde{B}_\alpha)_{1 \leq \alpha < \omega_1}$  of subsets of  $\mathbb{R}^\omega$  in the following way. Put

$$\tilde{A}_1 = \{x = (x_n)_{n=1}^\infty \in \mathbb{R}^\omega : \exists m \forall n \geq m \ x_n = 0\} \quad \text{and} \quad B_1 = \mathbb{R}^\omega \setminus A_1.$$

Let  $\mathbb{N} = \bigcup_{n=1}^\infty T_n$  be a union of a sequence of mutually disjoint infinite sets  $T_n = \{t_{n1}, t_{n2}, \dots\}$ , where  $(t_{nm})_{m=1}^\infty$  is a strictly increasing sequence of numbers  $t_{nm} \in \mathbb{N}$ . For every  $n \in \mathbb{N}$  we denote by  $\tilde{A}_1^n / \tilde{B}_1^n$  the copy of the set  $\tilde{A}_1 / B_1$ , which is contained in the space  $\mathbb{R}^{T_n}$ . Assume that for some  $\alpha \geq 1$  we have already defined sequences  $(\tilde{A}_\beta)_{1 \leq \beta < \alpha}$  and  $(\tilde{B}_\beta)_{1 \leq \beta < \alpha}$  (and their copies  $(\tilde{A}_\beta^n)_{1 \leq \beta < \alpha}$  and  $(\tilde{B}_\beta^n)_{1 \leq \beta < \alpha}$  in  $\mathbb{R}^{T_n}$ ) of subsets of  $\mathbb{R}^\omega$ . Now we put

$$\tilde{A}_\alpha = \begin{cases} \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty \pi_{T_n}^{-1}(\tilde{B}_\beta^n), & \alpha = \beta + 1, \\ \bigcup_{n=1}^\infty \pi_{T_n}^{-1}(\tilde{A}_{\beta_n}^n), & \alpha = \sup \beta_n, \end{cases}$$

$$\tilde{B}_\alpha = \mathbb{R}^\omega \setminus \tilde{A}_\alpha.$$

Let for every  $\alpha \in [1, \omega_1)$

$$A_\alpha = \tilde{A}_\alpha \cap \ell_p, \quad B_\alpha = \tilde{B}_\alpha \cap \ell_p,$$

$$A_\alpha^n = \tilde{A}_\alpha^n \cap \ell_p(\mathbb{R}^{T_n}), \quad B_\alpha^n = \tilde{B}_\alpha^n \cap \ell_p(\mathbb{R}^{T_n}).$$

**CLAIM 1.** For every  $\alpha \in [1, \omega_1)$  the sets  $A_\alpha$  and  $B_\alpha$  are  $\mathcal{S}$ -open in  $\ell_p$ .

*Proof of Claim 1.* Evidently,  $A_1$  and  $B_1$  are  $\mathcal{S}$ -open. Assume that for some  $\alpha < \omega_1$  the claim is valid for all  $\beta < \alpha$ . Let  $\alpha = \beta + 1$  be an isolated ordinal. Take any  $x \in A_\alpha$  and  $y \in \sigma_1(x)$ . Then there exists  $m \in \mathbb{N}$  such that  $\pi_{T_n}(x) \in B_\beta^n$  for all  $n \geq m$ . Since  $\pi_{T_n}(y) \in \sigma_1(\pi_{T_n}(x))$  and  $B_\beta^n$  is  $\mathcal{S}$ -open,  $\pi_{T_n}(y) \in B_\beta^n$ . Therefore,  $y \in A_\alpha$ . We argue similarly in the case where  $\alpha$  is a limit ordinal.  $\square$

Consider the equivalent metric

$$d(x, y) = \min\{d_p(x, y), 1\}$$

on the space  $\ell_p$ .

CLAIM 2. For every  $\alpha \in [1, \omega_1)$  the following condition holds:

(\*) for every set  $C \subseteq (\ell_p, d)$  of the additive /multiplicative/ class  $\alpha$  there exists a contracting mapping  $f : (\ell_p, d) \rightarrow (\ell_p, d)$  with the Lipschitz constant  $q = \frac{1}{2}$  such that

$$C = f^{-1}(A_\alpha) \quad /C = f^{-1}(B_\alpha)/, \quad (3.1)$$

$$|\pi_n(f(x))| \leq 1 \quad \forall x \in \ell_p \quad \forall n \in \mathbb{N}. \quad (3.2)$$

*Proof of Claim 2.* We will argue by the induction on  $\alpha$ . Let  $\alpha = 1$  and  $C$  be an arbitrary  $F_\sigma$ -subset of  $(\ell_p, d)$ . Then  $C = \bigcup_{n=1}^{\infty} C_n$  is a union of an increasing sequence of closed sets  $C_n \subseteq (\ell_p, d)$ . For every  $x \in \ell_p$  we put

$$f(x) = \left(\frac{1}{3}d(x, C_1), \dots, \frac{1}{3^n}d(x, C_n), \dots\right).$$

Since every  $d(x, C_n) \leq 1$ ,  $f(x) \in \ell_p$  for every  $x \in \ell_p$ . Show that  $C = f^{-1}(A_1)$ . Take  $x \in C$  and choose  $m \in \mathbb{N}$  such that  $x \in C_n$  for all  $n \geq m$ . Then  $d(x, C_n) = 0$  and  $\pi_n(f(x)) = 0$  for all  $n \geq m$ . Hence,  $x \in f^{-1}(A_1)$ . The inverse inclusion follows from the closedness of  $C_n$ . Since

$$d(f(x), f(y)) \leq d_p(f(x), f(y)) = \left(\sum_{n=1}^{\infty} \frac{1}{3^{np}} |d(x, C_n) - d(y, C_n)|^p\right)^{\frac{1}{p}} \leq d(x, y) \left(\sum_{n=1}^{\infty} \frac{1}{3^{np}}\right)^{\frac{1}{p}} \leq \frac{1}{2}d(x, y)$$

for all  $x, y \in \ell_p$ , the mapping  $f : (\ell_p, d) \rightarrow (\ell_p, d)$  is contracting with the Lipschitz constant  $q = \frac{1}{2}$ . Moreover,  $|\pi_n(f(x))| = \frac{1}{3^n}d(x, C_n) \leq 1$  for every  $n \in \mathbb{N}$ .

Assume that for some  $\alpha < \omega_1$  the condition (\*) is valid for all  $\beta < \alpha$ . Let  $C \subseteq (\ell_p, d)$  be any set of the  $\alpha$ 'th additive class. Take an increasing sequence of sets  $C_n$  such that  $C = \bigcup_{n=1}^{\infty} C_n$ , where every  $C_n$  belongs to the multiplicative class  $\beta$  if  $\alpha = \beta + 1$ , and in the case  $\alpha = \sup \beta_n$  we can assume that  $C_n$  belongs to the additive class  $\beta_n$  for every  $n \in \mathbb{N}$ . By the inductive assumption for every  $n \in \mathbb{N}$  there exists a contracting mapping  $f_n : (\ell_p, d) \rightarrow (\ell_p, d)$  with the Lipschitz constant  $q = \frac{1}{2}$  such that

$$C_n = \begin{cases} f_n^{-1}(B_\beta), & \alpha = \beta + 1, \\ f_n^{-1}(A_{\beta_n}), & \alpha = \sup \beta_n, \end{cases} \quad (3.3)$$

$$|\pi_m(f_n(x))| = |f_{nm}(x)| \leq 1 \quad \forall x \in \ell_p \quad \forall n, m \in \mathbb{N}. \quad (3.4)$$

For every  $k \in \mathbb{N}$  we choose a unique pair  $(n(k), m(k)) \in \mathbb{N}^2$  such that

$$k = t_{n(k)m(k)} \in T_{n(k)}.$$

For every  $x \in \ell_p$  we put

$$f(x) = \left(\frac{1}{3^2}f_{n(1)m(1)}(x), \dots, \frac{1}{3^{k+1}}f_{n(k)m(k)}(x), \dots\right).$$

It is easy to see that  $f(x) \in \ell_p$  for every  $x \in \ell_p$ .

Since

$$\frac{1}{3}|f_{nm}(x) - f_{nm}(y)| \leq d_p(f_n(x), f_n(y)) \quad \text{and} \quad \frac{1}{3}|f_{nm}(x) - f_{nm}(y)| \leq \frac{2}{3} \leq 1,$$

we have

$$\frac{1}{3}|f_{nm}(x) - f_{nm}(y)| \leq d(f_n(x), f_n(y)) \leq \frac{1}{2}d(x, y)$$

for all  $x, y \in \ell_p$  and  $n, m \in \mathbb{N}$ . Consequently,

$$\begin{aligned} d(f(x), f(y)) &\leq d_p(f(x), f(y)) = \left( \sum_{k=1}^{\infty} \frac{1}{3^{kp}} \left( \frac{1}{3} |f_{n(k)m(k)}(x) - f_{n(k)m(k)}(y)| \right)^p \right)^{\frac{1}{p}} \leq \\ &\leq \frac{1}{2} d(x, y) \left( \sum_{k=1}^{\infty} \frac{1}{3^{kp}} \right)^{\frac{1}{p}} \leq \frac{1}{2} d(x, y) \end{aligned}$$

for all  $x, y \in \ell_p$ . Therefore,  $f$  has the Lipschitz constant  $q = \frac{1}{2}$ .

Finally, it is easy to verify that  $C = f^{-1}(A_\alpha)$ .  $\square$

**CLAIM 3.** For every  $\alpha \in [1, \omega_1)$  the set  $A_\alpha$  belongs to the additive class  $\alpha$  and does not belong to the multiplicative class  $\alpha$  in  $\ell_p$ .

*Proof of Claim 3.* We first prove that the set  $\tilde{A}_\alpha / \tilde{B}_\alpha /$  is of the  $\alpha$ 'th additive /multiplicative/ class in  $\mathbb{R}^\omega$ .

If  $\alpha = 1$ , then  $\tilde{A}_1 = \bigcup_{n=1}^{\infty} \sigma_n(0)$  is an  $F_\sigma$ -subset of  $\mathbb{R}^\omega$ , since every  $\sigma_n(0)$  is closed in  $\mathbb{R}^\omega$ . Consequently,  $\tilde{B}_1$  is  $G_\delta$  in  $\mathbb{R}^\omega$ . Suppose that for some  $\alpha \geq 1$  the set  $\tilde{A}_\beta / \tilde{B}_\beta /$  belongs to the additive /multiplicative/ class  $\beta$  in  $\mathbb{R}^\omega$  for every  $\beta < \alpha$ . Since every projection  $\pi_{T_n} : \mathbb{R}^\omega \rightarrow \mathbb{R}^{T_n}$  is continuous, the set  $\tilde{A}_\alpha$  belongs to the additive class  $\alpha$  in  $\mathbb{R}^\omega$  and the set  $\tilde{B}_\alpha$  belongs to the multiplicative class  $\alpha$  in  $\mathbb{R}^\omega$ .

Since the topology of pointwise convergence on  $\ell_p$  is weaker than the topology generated by the norm  $\|\cdot\|_p$ , for every  $\alpha$  the set  $A_\alpha / B_\alpha /$  is of the  $\alpha$ 'th additive /multiplicative/ class in  $\ell_p$ .

Fix  $\alpha \in [1, \omega_1)$ . In order to show that  $A_\alpha$  does not belong to the  $\alpha$ 'th multiplicative class we assume the contrary. Then there exists a contracting mapping  $f : (\ell_p, d) \rightarrow (\ell_p, d)$  such that  $A_\alpha = f^{-1}(B_\alpha)$ . By the Contraction Map Principle, there exists a fixed point for the mapping  $f$ , which implies a contradiction.

It remains to put  $E = A_\alpha$ .  $\square$

**Theorem 3.2.** Let  $\alpha \in [1, \omega_1)$  and  $p \in [1, +\infty)$ . Then there exists an ssc-function  $f : \ell_p \rightarrow \mathbb{R}$  which belongs to the  $(\alpha + 1)$ 'th Baire class and does not belong to the  $\alpha$ 'th Baire class.

*Proof.* By Lemma 3.1 there exists an  $\mathcal{S}$ -open set  $E$  in  $\ell_p$  such that  $E$  belongs to the  $\alpha$ 'th additive class and does not belong to the  $\alpha$ 'th multiplicative class. Then the function  $f : \ell_p \rightarrow \mathbb{R}$ ,

$$f(x) = \chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E, \end{cases}$$

satisfies the required properties.  $\square$

The existence of an ssc-function  $f : \mathbb{R}^\omega \rightarrow \mathbb{R}$  which is not Baire measurable was proved in [5, Proposition 3.2].

#### 4. DISCONTINUITIES OF SSC-FUNCTIONS ON $\ell_p$

We denote by  $\ell_p^\omega$  the set  $\ell_p \subseteq \mathbb{R}^\omega$  endowed with the topology of pointwise convergence induced from  $\mathbb{R}^\omega$ . Evidently,  $\text{SSC}(\ell_p^\omega) \subseteq \text{SSC}(\ell_p)$ . The converse is not true as the following example shows.

*Example 1.* There exists a continuous function  $f : \ell_2 \rightarrow \mathbb{R}$  which is not strongly separately continuous on  $\ell_2^\omega$ .

*Proof.* For every  $n \in \mathbb{N}$  we set

$$x_n = \underbrace{\left(\frac{1}{n}, 0, \dots, 0, n, 0, \dots\right)}_{n-1} \text{ and } y_n = \underbrace{(0, \dots, 0, n, 0, \dots)}_{n-1}.$$

Since the sets  $F_1 = \{x_n : n \in \mathbb{N}\}$  and  $F_2 = \{y_n : n \in \mathbb{N}\}$  are disjoint and closed in  $\ell_2$ , the function  $f : \ell_2 \rightarrow \mathbb{R}$  defined by the formula

$$f(x) = \frac{d_2(x, F_1)}{d_2(x, F_1) + d_2(x, F_2)}$$

is continuous and  $F_1 = f^{-1}(0)$ ,  $F_2 = f^{-1}(1)$ . Notice that  $x_n \rightarrow 0$  in  $\ell_2^\omega$  and  $y_n = (x_n)_1^0$ . But  $f(x_n) - f(y_n) = 1$  for every  $n$ , which implies that  $f$  is not strongly separately continuous on  $\ell_2^\omega$  at the point  $x = 0$  with respect to the first variable.  $\square$

By  $C(f)$  ( $D(f)$ ) we denote the set of all points of continuity (discontinuity) of a mapping  $f$ .

**Theorem 4.1.** *For any open nonempty set  $G \subseteq \ell_p$  with  $1 \leq p < \infty$  there exists a strongly separately continuous function  $f : \ell_p \rightarrow \mathbb{R}$  such that  $D(f) = G$ .*

*Proof.* Fix  $p \in [1, +\infty)$  and let  $\sigma = \sigma(0) = \{(x_n)_{n=1}^\infty \in \ell_p : (\exists k \in \mathbb{N}) (\forall n \geq k) (x_n = 0)\}$ . Denote  $F = \ell_p \setminus G$ . For every  $x = (x_n)_{n \in \mathbb{N}} \in \ell_p$  we put

$$\varphi(x) = \begin{cases} \min\{d_p(x, F), 1\}, & F \neq \emptyset, \\ 1, & F = \emptyset, \end{cases}$$

$$g(x) = \begin{cases} \exp(-\sum_{n=1}^\infty |x_n|), & x = (x_n)_{n=1}^\infty \in \sigma, \\ 1, & x \in \ell_p \setminus \sigma, \end{cases}$$

and let

$$f(x) = \varphi(x) \cdot g(x).$$

CLAIM 1.  $F \subseteq C(f)$ .

*Proof.* Fix  $x^0 \in F$  and take any convergent sequence  $(x^m)_{m=1}^\infty$  to  $x^0$  in  $\ell_p$ . Notice that

$$\lim_{m \rightarrow \infty} \varphi(x^m)g(x^m) = 0,$$

because  $\varphi(x)$  is continuous at  $x^0$  and  $g(x)$  is bounded. Then

$$\lim_{m \rightarrow \infty} f(x^m) = 0 = f(x^0).$$

Hence,  $x^0 \in C(f)$ .

CLAIM 2.  $G \subseteq D(f)$ .

*Proof.* Fix  $x^0 \in G$ . Then  $f(x^0) > 0$ . We put  $\varepsilon = \frac{1}{2}f(x^0)$  and take an arbitrary  $\delta > 0$ . Since the set  $D = \ell_p \setminus \sigma$  is dense in  $\ell_p$ , there exists  $x = (x_n)_{n \in \mathbb{N}} \in \ell_p$  such that

$$\|x - x^0\|_p < \frac{\delta}{2} \text{ and } x \notin \sigma.$$

Take a number  $N$  such that

$$\sum_{n=1}^N |x_n| > \ln\left(\frac{1}{f(x^0) - \varepsilon}\right) \text{ and } \sum_{n=N+1}^\infty |x_n|^p < \left(\frac{\delta}{2}\right)^p.$$

We put

$$y = (x_1, \dots, x_N, 0, 0, \dots).$$

Then  $y \in \sigma$  and

$$\|y - x^0\|_p \leq \|y - x\|_p + \|x - x^0\|_p = \left(\sum_{n=N+1}^\infty |x_n|^p\right)^{\frac{1}{p}} + \|x - x^0\|_p < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

But

$$f(x^0) - f(y) = f(x^0) - \varphi(y) \cdot \exp\left(-\sum_{n=1}^N |x_n|\right) > f(x^0) - \exp\left(-\sum_{n=1}^N |x_n|\right) > f(x^0) + \varepsilon - f(x^0) = \varepsilon,$$

which implies that  $f$  is discontinuous at  $x^0$ .

CLAIM 3.  $f : \ell_p \rightarrow \mathbb{R}$  is strongly separately continuous.

*Proof.* Let  $x^0 \in \ell_p$ . Evidently,  $f$  is strongly separately continuous at  $x^0$  if  $x^0 \in F$ . Therefore, we assume that  $x^0 \in G$ . Fix  $k \in \mathbb{N}$  and  $\varepsilon > 0$ . Take  $\delta_1 > 0$  with  $B(x^0, \delta_1) \subseteq G$ . Since  $\varphi(x)$  is continuous at  $x^0$ , there exists  $\delta_2 > 0$  such that

$$|\varphi(x) - \varphi(x^0)| < \frac{\varepsilon}{4}$$

for all  $x \in B(x^0, \delta_2)$ . Put

$$\delta = \min\{\delta_1, \delta_2, \ln(1 + \frac{\varepsilon}{2})\}.$$

Now let  $x \in B(x^0, \delta)$  and  $y = x_k^{x^0} \in B(x^0, \delta)$ . If  $x \notin \sigma$ , then  $y \notin \sigma$ . In this case

$$|f(x) - f(y)| = |\varphi(x) - \varphi(y)| \leq |\varphi(x) - \varphi(x^0)| + |\varphi(x^0) - \varphi(y)| < \varepsilon.$$

Assume that  $x \in \sigma$ . Then  $y \in \sigma$  and

$$\begin{aligned} |f(x) - f(y)| &\leq |g(x)| |\varphi(x) - \varphi(y)| + |\varphi(y)| |g(x) - g(y)| < \\ &< \frac{\varepsilon}{2} + |\exp(-\sum_{n=1}^{\infty} |x_n|) - \exp(-\sum_{n=1}^{\infty} |y_n|)| < \frac{\varepsilon}{2} + |\exp(\sum_{n=1}^{\infty} |y_n| - \sum_{n=1}^{\infty} |x_n|) - 1|. \end{aligned}$$

Taking into account that

$$\exp(-\sum_{n=1}^{\infty} |x_n - y_n|) \leq \exp(\sum_{n=1}^{\infty} |x_n| - \sum_{n=1}^{\infty} |y_n|) \leq \exp(\sum_{n=1}^{\infty} |x_n - y_n|),$$

we obtain that

$$|f(x) - f(y)| < \frac{\varepsilon}{2} + \exp(|x_k - x_k^0|) - 1 < \frac{\varepsilon}{2} + \exp(\delta) - 1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence,  $f$  is strongly separately continuous at  $x^0$  with respect to the  $k$ 'th variable.  $\square$

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